UMBRAL CALCULUS

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ABSTRACT. Umbral calculus is a long-studied theory in combinatorics that provides methods for representing sequences and can lead to interesting results. We introduce the history of umbral calculus, from its shaky beginnings to its classical applications and finally the sophisticated algebra used to formalize it. We focus on classic results with important polynomials defined by generating functions, giving rise to identities for notable sequences and combinatorical structures like the Bernoulli and Bell numbers. We also cover the concepts of linear functionals used to ground umbral calculus in hard algebra, and show a proof of the Lagrange Inversion Theorem using it.

1. Introduction to Umbral Calculus

Umbral calculus is from its beginnings a very surprising and fascinating field of study. At its simplest, it is a method of representing series of the form $\sum a_n x_n/n!$, exponential generating functions, and working with them to derive interesting new relations that were not possible otherwise. The word "umbra", meaning "shadow" in Latin, is used to denote the projection of the power onto the index of a term in a sequence: a notational trick at first but now a well-supported theory. In some sense, though, it can also be interpreted as reflecting the shadowy mysteriousness of this method at first glance!

Some of the key ideas, like some polynomial sequences x_n (e.g. the falling factorial) mirroring the power sequence x^n, go back to Barrow and Newton in the 17th century. Real formal inquiry was only done in the late 19th century, though, starting with Cayley, Blissard, and Sylvester (who invented the name umbral calculus). Still, umbral calculus was mostly a magic method, without an axiomatic basis. The work of Steven Roman and Gian-Carlo Rota, among others, finally developed umbral calculus into a formal theory. They used the idea of working with linear functionals on a vector space of polynomials, mapping them to the elements of a relevant sequence.

Some later works on umbral calculus have further abstracted the notation to focus on algebraic developments in a very wide field. The language of umbral calculus based on functionals applied to polynomials allows everything from Mittag–Leffler functions to trigonometry to be reinterpreted. We will mostly focus on key classical results working with polynomials defined by exponential generating functions, which give us interesting findings in polynomial sequences and combinatorical identities. At the end, a proof of the Lagrange inversion theorem without the need for complex analysis shows the versatility of umbral calculus's functionals.

2. Early Umbral Calculus

For a prelude, here is an example from [1] of some of the early "magical" relationships between sequences and certain powers that led to the development of umbral calculus: In

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the case of the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$, we note that just like x^n counts the number of functions from an n-element set to an x-element set, $(x)_n$ counts the number of injections. As the derivative maps x^n to nx^{n-1}, the forward difference (written as $\Delta f(x) = f(x+1) - f(x)$) maps $(x)_n$ to $n(x)_{n-1}$. Also, polynomials can be expressed in terms of x^n via Taylor's theorem

$$f(x+a) = \sum_{n=0}^{\infty} a^n \frac{f^{(n)}(x)}{n!}$$

while Newton's theorem similarly gives for $(x)_n$

$$f(x+a) = \sum_{n=0}^{\infty} (a)_n \frac{\Delta^n f(x)}{n!}.$$

And just as $(x+y)^n$ can be expanded using the binomial theorem as

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k},$$

$(x+y)_n$ can be expanded with the Chu-Vandermonde identity ($\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$) to give

$$(x+a)_n = \sum_{k=0}^{\infty} \binom{n}{k} (a)_k (x)_{n-k}.$$

These sorts of connections were assuredly fascinating, but for the longest time lacked much basis in proof!

3. The Linear Functional Basis

As mentioned in the introduction, umbral calculus was considered to be a sort of magic rule: it could lead to results by raising and lowering some indices in equations, but the actual foundational basis for it was weak. Putting it on solid ground required the idea, introduced by Rota in [6], of a *linear functional*:

Definition 3.1 (Linear functional)**.** A linear functional, or linear form, is a linear map from a vector space to its field of scalars.

Here, we look at mappings from sequences of polynomials x^n onto related sequences a_n. We call this a linear functional L, and write it like the bra-ket notation used in physics as

$$\langle L \mid x^n \rangle = a_n.$$

This linear functional L is what was called an umbra by the early explorers of umbral calculus, but back then it was less a rigorous algebra and more a haphazard rule.

Note that two linear functionals L and M are equal if and only if

$$\langle L \mid p_n(x) \rangle = \langle M \mid p_n(x) \rangle$$

for all $p_n(x)$ in a polynomial sequence. We call this the *spanning argument* and will refer back to it in later proofs.

For the sake of notation, we have P denote the algebra of all polynomials in x, and $P*$ the vector space of all linear functionals on P. We will also use *linear operators*, which map a vector space to itself.

Definition 3.2 (Delta functional)**.** A delta functional is a linear functional L that satisfies $\langle L \mid 1\rangle = 0$ and $\langle L \mid x\rangle \neq 0$.

We can associate a formal power series to any linear functional M given a delta functional L: if

$$M = \sum_{k=0}^{\infty} a_k L^k,$$

we associate to M the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k,$$

which is called the *indicator* of M.

We define the *conjugate sequence* of a delta functional L as the polynomial sequence

$$q_n(x) = \sum_{k>0} \frac{\langle L^k \mid x^n\rangle}{k!} x^k.$$

This sequence is said to be the reciprocal of the associated sequence of the delta functional.

Setting up these formal constructs will enable us to give rigorous justification for our results that follow.

4. Sheffer Polynomials

Umbral calculus is a powerful way to work with polynomial sequences. Much of the study of umbral calculus deals with a certain class of these sequences, named after Isador Sheffer.

Recall that earlier we mentioned analogies between the falling factorial sequence and the powers of x. In fact, the falling factorial sequence can be seen to be a subset of a class of sequences satisfying $p_n' = np_{n-1}$, known as Appell sequences (or sequences of Appell polynomials) after Paul Appell who studied them in the 19th century.

Definition 4.1 (Appell sequence)**.** An Appell sequence is any sequence of polynomials denoted $p_n(x)$, where $p_0(x)$ is a non-zero constant and

$$\frac{d}{dx} p_n(x) = np_{n-1}(x)$$

In 1939, Sheffer noticed the umbral relationships we have described, and based on these ideas, Sheffer extended these into what are now called Sheffer sequences.

Definition 4.2 (Sheffer sequence)**.** A polynomial sequence $s_n(x)$ is a Sheffer sequence relative to a sequence $p_n(x)$ of binomial type if it satisfies the functional equation

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y)$$

for all $n \geqslant 0$ and for all $y \in K$.

Equivalently, a polynomial sequence $s_n(x)$ is a Sheffer sequence if and only if there exist a sequence of binomial type $p_n(x)$ and an invertible shift-invariant operator P such that

$$p_n(x) = Ps_n(x)$$

for all $n \geqslant 0$. Shift-invariance here simply means shifting (translating) the inputs shifts the outputs by the same amount, and swapping the order of the operator with a translation does not change the result (i.e. it commutes with translation operators).

As an example, we will present one generalized identity for Appell polynomials found in [3]. More detailed investigations of Sheffer sequences can be found in [4] and [7].

Proposition 4.3. *Let m, n be non-negative integers. Then*

$$\sum_{k=0}^{n} \binom{n}{k} y^{n-k} p_{m+k}(x) = \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} p_{n+k}(x+y).$$

Proof. We can let A^n be the umbra $A^n = a_n$. Then from the definition of Appell sequences we can obtain the power series form

$$\sum_{n\geq 0} p_n(x) \frac{t^n}{n!} = e^{(A+x)t},$$

giving the umbral representation $p_n(x) = (A+x)^n$. Then

$$\begin{aligned}
(A+x+y)^n(A+x)^m &= (A+x+y)^n(A+x+y-y)^m \\
&= \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} (A+x+y)^{n+k} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} p_{n+k}(x+y),
\end{aligned}$$

and on the other hand,

$$(A+x+y)^n(A+x)^m = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} (A+x)^{m+k} = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} p_{m+k}(x).$$

Equating the two expressions for $(A+x+y)^n(A+x)^m$ gives the identity. $\square$

5. Bernoulli Numbers

Here are some applications of umbral calculus to Bernoulli numbers. Mainly "classical" methods are used, with less linear functional algebra.

Definition 5.1 (Bernoulli numbers)**.** The Bernoulli numbers appear in a variety of contexts and have recursive and explicit forms. Here we give the exponential generation function:

$$B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}$$

Finding these numbers are one of the quintessential examples of umbral calculus in practice. Using an umbral "magic trick" found in [1] they can be derived as

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \simeq \sum_{n=0}^{\infty} B^n \frac{x^n}{n!} = e^{Bx},$$

then

$$e^{(B+1)x} - e^{Bx} = x,$$

after which looking at coefficients in the form $\frac{x^n}{n!}$ gives

$$(B+1)^n - B^n \simeq \begin{cases} 0 & \text{if } n \neq 1, \\ 1 & \text{if } n = 1. \end{cases}$$

Expanding this using the Binomial Theorem and changing the superscript powers back to subscript indices, we can obtain the relation

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \begin{cases} 0 & \text{if } n \neq 1, \\ 1 & \text{if } n = 1. \end{cases}$$

A standard proof can be done by looking at the power series $\left(e^x - 1\right)/x)$.

We give a more formal presentation found in [2], based on the work by Rota and Taylor in [5]. The Bernoulli numbers B_n are defined by the exponential generating function

$$B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

Since this implies that $e^x B(x) = x + B(x)$, the Bernoulli umbra B defined by $B^n = B_n$ satisfies

$$(B+1)^n = B^n + \delta_{n-1},$$

where δ_m is 1 if $m = 0$ and is 0 otherwise. (This notation is known as the Kronecker delta and is a somewhat more convenient alternative to the cases used earlier.) Then by linearity we see that for any admissible formal power series f,

$$f(B+1) = f(B) + f'(0). \tag{$*$}$$

Iterated this gives

$$f(B+k) = f(B) + f'(0) + f'(1) + \cdots + f'(k-1)$$

for any nonnegative integer k, and we are done.

There are three other important basic identities for the Bernoulli umbra.

Theorem 5.2. *The following are true:*

(i) $(B+1)^n = (-B)^n$.

(ii) $(-B)^n = B^n$ *for* $n \neq 1$, *with* $B_1 = -\frac{1}{2}$. *Thus* $B_n = 0$ *when* n *is odd and greater than* *1* .

(iii) *For any positive integer* k,

$$kB^n = (kB)^n + (kB+1)^n + \cdots + (kB+k-1)^n.$$

Proof. We prove "linearized" versions of these formulas: for any polynomial f, we have

$$\begin{aligned} f(B+1) &= f(-B) \\ f(-B) &= f(B) + f'(0) \\ kf(B) &= f(kB) + f(kB+1) + \cdots + f(kB+k-1). \end{aligned}$$

The second equation is a direct consequence of the first equation and $(*)$. We prove the other two by choosing polynomials so that they follow from $(*)$ as well.

For the first equation, we take $f(x) = x^n - (x-1)^n$, where $n \geq 1$. Then

$$f(B+1) = (B+1)^n - B^n = \delta_{n-1},$$

and since $f(-x) = (-1)^{n-1} f(x+1)$, we have

$$f(-B) = (-1)^{n-1} f(B+1) = (-1)^{n-1}\delta_{n-1} = \delta_{n-1} = f(B+1).$$

For the third equation, we take $f(x) = (x+1)^n - x^n$, where $n \geq 1$. Then $f(B) = \delta_{n-1}$ and

$$\begin{aligned}\sum_{i=0}^{k-1} f(kB+i) &= \sum_{i=0}^{k-1}(kB+i+1)^n - \sum_{i=0}^{k-1}(kB+1)^n \\ &= (kB+k)^n - (kB)^n = k^n\left((B+1)^n - B^n\right) \\ &= k^n\delta_{n-1} = k\delta_{n-1} = kf(B).\end{aligned}$$

□

6. Combinatoric Identities

Since the classic formulation of umbral calculus is based on generating functions, one may expect that its techniques may be applied to a range of combinatorical problems. Indeed, elegant relations for important sequences like the Bell numbers and powerful arguments for counting problems can be derived with the power of the umbra.

Definition 6.1 (Bell numbers)**.** B_n is the number of ways to partition a set with length n. A simple expression among many is the exponential generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}.$$

(Note: Here B is used to denote the Bell numbers, not Bernoulli numbers as done earlier.)

This can be proven in multiple ways, including analytical combinatorics, but we will give a proof of it along with multiple other exciting relations based only on umbral calculus. The source is [6], where some of the linear functional theory was first presented.

We consider an finite set U with $u > 0$ elements. We look at the structure of the set U^S of functions with domain S, a set with n elements, and range a subset of U. It is simple to find that there are u^n distinct such functions. Now in further detail:

Every function $f : S \to U$ has an associated partition π of the set S, called the kernel of f, defined so that two elements a and b of S belong to the same block of π if and only if $f(a) = f(b)$.

How many distinct functions are there with a given kernel π?To find this, we let $N(\pi)$ be the number of distinct blocks of the partition π. A function having kernel π has to take distinct values on distinct blocks of π. Therefore, such a function takes $N(\pi)$ distinct values, and the number of distinct such functions is equal to the number of one-to-one functions from a set of $N(\pi)$ elements to the set U. It is also easy to find that number as being $u(u-1)\cdots(u-N(\pi)+1) = (u)_{N(\pi)}$. (This is the falling factorial of u with exponent $N(\pi)$.)

Every function has a unique kernel, so we have for all integers $u > 0$

$$(*) \qquad \sum_{\pi} (u)_{N(x)} = u^n,$$

where the sum on the left ranges over all partitions π of the set S.

Now onto the main idea: let V be the vector space over the reals consisting of all polynomials in u. Any sequence of polynomials of degrees $0, 1, 2, \cdots$, is a basis for this vector space — in particular, the sequence $(u)_0 = 1, (u)_1, (u)_2, (u)_3, \cdots$. Since a linear functional L

on V is uniquely determined by arbitrary assigning the values it takes, there exists a unique linear functional L on V such that

$$L(1) = 1, \quad L\left((u)_k\right) = 1, \quad k = 1, 2, 3, \cdots .$$

Applying L to both sides of $(*)$ we obtain

$$\sum_x L\left((u)_{N(x)}\right) = L\left(u^n\right),$$

but, by the definition of L, the left side simplifies to a sum of as many ones as there are partitions of the set S. In other words, we may simplify the above to

$$B_n = L\left(u^n\right).$$

This is the explicit expression for the Bell numbers we want. From this, we can get more useful properties.

We start by deriving the recursion formula for the numbers B_n,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Now, since $u(u-1)_n = (u)_{n+1}$, we have $L\left(u(u-1)_n\right) = 1 = L\left((u)_n\right)$. Since the polynomials $1, (u)_n$ for $n = 2, 3, \cdots$ form a basis for the vector space V, it follows from the linearity of L, that

$$L(up(u-1)) = L(p(u))$$

for every polynomial p. Specifically, for $p(u) = (u+1)^n$ we obtain

$$L\left(u^{n+1}\right) = L\left((u+1)^n\right),$$

which is the recursion formula we want.

We can derive the generating function directly from our explicit expression as well. We have

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{L\left(u^n\right)}{n!} x^n = L\left(e^{ux}\right).$$

Then, setting $e^z = 1 + v$ and expanding $(1+v)^u$ by the binomial theorem gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n &= L\left((1+v)^u\right) = L\left(\sum_{n=0}^{\infty} \frac{(u)_n}{n!} v^n\right) = \sum_{n=0}^{\infty} \frac{L\left((u)_n\right)}{n!} v^n \\ &= e^v = e^{e^x - 1}. \end{aligned}$$

It is possible to do the above proof without the linear functional L, but it is simpler to do so with it, as with many other classical umbral calculus results.

The next result we will look at is by way of [4], and features heavy use of linear functional algebra to give another excellent proof of Cayley's formula.

Theorem 6.2 (Cayley's formula)**.** *The number of rooted, labeled trees on n vertices is n^{n-1}.*

Proof. Following the terminology Rota used, we define a store σ as a set, in general infinite, together with a map d that assigns every element of σ a positive integer, called its degree. The subset of σ consisting of all elements of a given degree is assumed to be finite. Here the elements of σ are structures and we want to count σ, i.e. to determine the number a_n of elements of σ of degree n. We call a_n the counting sequence of σ, and we assume that

$a_1 > 0$. We also define the generating functional of σ as the delta functional L satisfying $\langle L \mid x^n \rangle = a_n$.

The partitional of a store σ is a second store part (σ) defined so that an element p of part (σ) is a set (not a sequence) of pairs $\{(B_1, s_1), \ldots, (B_k, s_k)\}$, where

(i) the B_i are the blocks of a partition of the set $\{1, 2, \ldots, n\}$, for some n (hence B_i is nonempty);
(ii) the s_i are elements of the store σ;
(iii) the degree of s_i equals the number of elements in B_i.

Every such element p, called a part of part (σ) has an associated degree $d(p)$ of p (the sum of the degrees of s_i), and a part number of p (the number of blocks).

The partitional part (σ) is obtained by letting n range over all positive integers. We let $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, and call it the counting sequence of the partitional. We set $b_{0,0} = 1$. Since $a_1 > 0$, we have $b_{n,n} > 0$ for all n.

Let σ be the store whose elements of degree n are all rooted labeled trees on n vertices. Then part (σ) is the set of forests. Letting $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, we have the recursion obtained by removing the root of a tree and counting the resulting forest:

$$b_{n,1} = n \sum_k b_{n-1,k}$$

In terms of the generating functional, this becomes

$$\begin{aligned} \langle L \mid x^n \rangle &= n \sum_k \frac{\langle L^k \mid x^{n-1} \rangle}{k!} = \langle e^L \mid Dx^n \rangle \\ &= \langle Ae^L \mid x^n \rangle \end{aligned}$$

and thus $L = Ae^L$. We seek the conjugate sequence for L. But $A = Le^{-L} = f(L)$ and so $L = f^{-1}(A)$ and $L = f(A) = Ae^{-A}$. Thus we see that L is the Abel functional and

$$\sum_k b_{n,k} x^k = x(x+n)^{n-1}$$

Therefore,

$$b_{n,k} = \binom{n-1}{k-1} n^{n-k}$$

and the numbers of rooted labeled trees on n vertices is n^{n-1}. □

Owing to the limited scope of this paper, we have only scratched the surface of what may be studied with umbral calculus in combinatorics alone. Suggested topics for further inquiry are the Nörlund polynomials which extend the Bernoulli numbers, the applications of umbral calculus to idemtities in Stirling numbers and Euler numbers, and more. The opportunities for applying umbral calculus are nearly as large as the field of combinatorics.

7. Lagrange Inversion Theorem

We conclude with a proof of the Lagrange Inversion Theorem, found in [4], which nicely illustrates the power of the modern theory of umbral calculus based on linear algebra.

We first recall the basics of linear operator adjoints. Let T be a linear operator mapping P into itself. The adjoint T^* of T is the operator mapping P^* into itself defined by

$$\langle T^*(L) \mid p(x)\rangle = \langle L \mid Tp(x)\rangle$$

for all $L \in P^*$ and all $p(x) \in P$. The adjoint T^* of a linear operator T on P exists and is continuous.

Theorem 7.1 (Expansion Theorem)**.** *Let M be a linear functional and let L be a delta functional with associated sequence $p_n(x)$. Then*

$$M = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x)\rangle}{k!} L^k.$$

Proof. The result follows from the spanning argument, noting that

$$\sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x)\rangle}{k!} \left\langle L^k \mid p_n(x)\right\rangle = \sum_{k=0}^{n} \frac{\langle M \mid p_k(x)\rangle}{k!} n!\delta_{n,k} = \langle M \mid p_n(x)\rangle .$$

□

Proposition 7.2. *The polynomial sequence $p_n(x)$ is the associated sequence for the delta operator Q if and only if it satisfies the following conditions: (i) $p_0(x) = 1$, (ii) $p_n(0) = 0$ for $n > 0$, (iii) $Qp_n(x) = np_{n-1}(x)$.*

Proof. Let $Q = \mu(L)$ and suppose first that $p_n(x)$ is the associated sequence for Q, and hence for L. Then

$$\begin{aligned}\left\langle L^k \mid Qp_n(x)\right\rangle &= \left\langle L^{k+1} \mid p_n(x)\right\rangle \\ &= n!\delta_{k+1,n} = \left\langle L^k \mid np_{n-1}(x)\right\rangle .\end{aligned}$$

Therefore, by the Expansion Theorem,

$$\langle M \mid Qp_n(x)\rangle = \langle M \mid np_{n-1}(x)\rangle$$

for every lincar functional M, and thus $Qp_n(x) = np_{n-1}(x)$. Conversely, suppose the polynomial sequence $p_n(x)$ satisfies (i), (ii), and (iii). Then

$$\begin{aligned}\left\langle L^k \mid p_n(x)\right\rangle &= \left\langle \epsilon \mid Q^k p_n(x)\right\rangle \\ &= \langle \epsilon \mid (n)_k p_{n-k}(x)\rangle = n!\delta_{n,k}\end{aligned}$$

so that $p_n(x)$ is the associated sequence for L. □

Theorem 7.3 (Transfer Formula)**.** *If $Q = PD$ is a delta operator, where P is an invertible shift-invariant operator, and if $p_n(x)$ is the associated sequence for Q, then for all $n \geqslant 0$,*

$$p_n(x) = Q' P^{-n-1} x^n$$

Proof. Letting $q_n(x) = Q'P^{-n-1}x^n$, we see that

$$Qq_n(x) = PDQ'P^{-n-1}x^n = nq_{n-1}(x)$$

and thus by (7.2) we need only show that $q_0(x) = 1$ and $q_n(0) = 0$ for $n > 0$. It is clear that $q_0(x)$ is a constant. Furthermore,

$$\begin{aligned}\langle \epsilon \mid q_0(x)\rangle &= \left\langle \epsilon \left| Q'P^{-1} \right| \right\rangle = \left\langle \epsilon \mid (P + DP')\, P^{-1} 1\right\rangle \\ &= \langle \epsilon \mid 1\rangle = 1\end{aligned}$$

and we have $q_0(x) = 1$. For $n > 0$,

$$\begin{aligned}
\langle \epsilon \mid q_n(x) \rangle &= \left\langle \epsilon \mid Q' p^{-n-1} x^n \right\rangle \\
&= \left\langle \epsilon \mid (P + DP')\, P^{-n-1} x^n \right\rangle \\
&= \langle \epsilon \mid P^n x^n \rangle + \left\langle \epsilon \mid nP' P^{-n1} x^{n-1} \right\rangle \\
&= \left\langle \epsilon \mid P^{-n} x^n \right\rangle - \left\langle \epsilon \mid \left(P^{-n}\right)' x^{n-1} \right\rangle \\
&= \left\langle \epsilon \mid P^{-n} x^n \right\rangle - \left\langle \mu^{-1} \left(P^{-n}\right) \mid x^n \right\rangle \\
&= 0
\end{aligned}$$

Thus $q_n(0) = 0$ for $n > 0$ and the theorem is proved. □

The Abel polynomials $A_n(x, a) = x(x - an)^{n-1}$ are the associated polynomials for the Abel functional $\epsilon_a A$, where

$$\langle \varepsilon_a A \mid p(x) \rangle = p'(a).$$

The indicator of the Abel functional is the series te^{at}.

Theorem 7.4 (Lagrange Inversion Theorem)**.** *Suppose z is defined as a function of w by an equation of the form*

$$z = f(w)$$

where f is analytic at a point a and $f'(a) \neq 0$. Then it is possible to invert or solve the equation for w, expressing it in the form $w = g(z)$ given by a power series,

$$g(z) = a + \sum_{n=1}^{\infty} g_n \frac{(z - f(a))^n}{n!}$$

where

$$g_n = \lim_{w \to a} \frac{d^{n-1}}{dw^{n-1}} \left[\left(\frac{w - a}{f(w) - f(a)} \right)^n \right]$$

Proof. The notation we have now makes Lagrange's inversion formula almost trivial. It states that if $f(t)$ is a delta series, then the nth coefficient in $f^{-1}(t)^k$ equals the $(n-k)$ th coefficient in $(f(t)/t)^{-n}$, multiplied by k/n. In our notation, this reads

$$\frac{\left\langle \tilde{L}^k \mid x^n \right\rangle}{n!} = \frac{k \left\langle M^{-n} \mid x^{n-k} \right\rangle}{n},$$

where the indicator of $L = AM$ is $f(t)$. The verification of this fact is now a trivial computation with adjoints. If $p_n(x)$ is the associated sequence for L, then using the Transfer Formula we find

$$\begin{aligned}
\left\langle \tilde{L}^k \mid x^n \right\rangle &= \left\langle A^k \mid p_n(x) \right\rangle = \left\langle A^k \mid x\mu(M)^{-n} x^{n-1} \right\rangle \\
&= \left\langle kA^{k-1} \mid \mu(M)^{-n} x^{n-1} \right\rangle = k \left\langle A^{k-1} M^{-n} \mid x^{n-1} \right\rangle \\
&= k \left\langle M^{-n} \mid D^{k-1} x^{n-1} \right\rangle = \frac{k}{n} (n)_k \left\langle M^{-n} \mid x^{n-k} \right\rangle
\end{aligned}$$

as desired. □

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