QUATERNIONIC ANALYSIS

ANDREW CHANG

Abstract. Quaternions are a four-dimensional algebra invented by William Rowan Hamilton that extend the complex numbers and are used abstract algebra, physics, and other fields. Although extending complex analysis to quaternions requires abandoning commutativity and equivalence between holomorphicity, analyticity, etc., many analogues to key ideas have been found, and even some interesting connections with modern physics. We give a survey of the work's basis and major results, beginning with the basics of quaternions and proceeding to a quaternion version of Cauchy's integral formula. At the end, we give a medley of applications.

1. Introduction to Quaternions

Quaternions are a four-dimensional algebra that extend the concept of complex numbers to form a new number system. They have been a subject of investigation in areas like abstract algebra, number theory, and applications to physics, mechanics, and computer modeling. Found by William Rowan Hamilton [5] in 1843, they were eventually displaced by vector algebra for the most part in physics and mathematics. However, quaternions still provide useful representations, are well researched algebraically, and in some cases are more suited for computations, leading to sustained interest from many fields.

Much work has been done to extend complex analysis into higher dimensions with quaternions, especially by the Swiss mathematician Fueter [4] (summarized in English by Deavours [1] and expanded by Sudbery [10]) with fair success. Although quaternions lose the important notions of the commutative property and the equivalence of holomorphicity, analyticity, etc., many analogues to complex analytic examples have been found, and even some interesting connections with modern physics. We summarize the work's basis and key results here, though some longer proofs are omitted.

The set of quaternions is represented by $\mathbb{H}$, and is formally a 4-dimensional vector space over $\mathbb{R}$, with a basis of $1, i, j, k$ where i, j, k are the basic quaternions (behaving much in the same way as the imaginary unit i).

Definition 1.1 (Quaternions)**.** A quaternion can be represented as a formal linear combination in the form $q = w + xi + yj + zk$, where $w, x, y, z \in \mathbb{R}$.

One might wonder whether it is possible to extend the quaternions to a similar algebra with three dimensions or even more than four. It is actually not possible to construct a number space that is a *real division algebra* (meaning addition/multiplication satisfy ring axioms, division is possible, scalar multiplication works) beyond the reals, complex numbers, and quaternions [3]. In fact, the Cayley–Dickson construction defines an

Date: May 18, 2021.

algebra similar to a direct sum, but with a different multiplication and a conjugation function, and allows for higher powers of two as dimensions, leading to the octonions (8) and sedenions (16), etc., but more and more fundamental properties are lost — notably, not even associativity works, and zero has multiple divisors. [9]

Definition 1.2. Multiplication between the basic quaternions can be summed up by the basic property that

$$i^2 = j^2 = k^2 = ijk = -1$$

and that each of i, j, k are anticommutative, meaning that

$$ij = -ji$$

and so on for any two of the three. Consequently, we also have

$$ij = k, jk = i, \text{ and } ki = j.$$

Sometimes in calculations we may abbreviate by writing the quaternions i, j, k as e_i for $i = 1, 2, 3$, and similarly the parts x, y, z as x_i for $i = 1, 2, 3$. We can thus use the Einstein summation convention to write, for example, a quaternion q as $w + x_i e_i$.

Definition 1.3. The conjugate of a quaternion q is

$$\overline{q} = w - xi - yj - zk.$$

Note that q commutes with $\overline{q}$ and their product is

$$q\overline{q} = w^2 + x^2 + y^2 + z^2.$$

We also define the modulus:

Definition 1.4. The modulus of a quaternion q is

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2} = \sqrt{q\overline{q}}.$$

Thus, the multiplicative inverse for non-zero quaternions is

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

The center of the algebra $\mathbb{H}$ is $\mathbb{R}$ (real numbers always commute). Also, for any quaternion q the vector space spanned by 1 and q is a subfield of the quaternions, and if 1 and q are linearly independent this subfield is isomorphic to $\mathbb{C}$. If we specifically consider the subfield spanned by 1 and i to be $\mathbb{C}$, then a quaternion can also be expressed as

$$q = (w + xi)1 + (y + zi)j,$$

since $ij = k$, or simply the ordered pair $(w + xi, y + zi)$ (essentially expressing a four-dimensional number as two two-dimensional numbers).

Quaternions can also be represented in various other forms, such as as a scalar part w and vector part $xi + yj + zj$, and as a 2×2 matrix

$$\begin{bmatrix} w + xi & y + zi \\ -y + zi & w - xi \end{bmatrix}.$$

The latter form has the advantage of quaternion addition and multiplication corresponding to matrix operations, the norm being related to the determinant, and other properties like a relationship with Pauli matrices for spin in quantum mechanics. [7]

2. Differential Forms

We use several differential forms to express the varying concepts of differntiability for quaternionic functions.

Definition 2.1. The gradient operator for a function $f : \mathbb{H} \to \mathbb{H}$ is defined by

$$\Box = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Definition 2.2. The Laplace operator for a function $f : \mathbb{H} \to \mathbb{H}$ is defined by

$$\Delta f = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Definition 2.3. The differential of a function $f : \mathbb{H} \to \mathbb{H}$ is defined as the 1-form

$$df = \frac{\partial f}{\partial w}dw + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

where w, x, y, z are the parts of a quaternion as previously defined. A function is then considered *(real-)differentiable* if this exists (in contrast to any other potential notions of differentiability for quaternions).

Definition 2.4. The differential at a point $q \in \mathbb{H}$ is then defined to be $df_q : \mathbb{H} \to \mathbb{H}$, giving a $\mathbb{R}$-linear map.

We also define the differential forms dq and Dq:

Definition 2.5. The differential of the identity function, dq, is defined as

$$dq = dw + idx + jdy + kdz.$$

Definition 2.6. Then we define Dq as a trilinear function so

$$\langle h_1, Dq\,(h_2, h_3, h_4)\rangle = v\,(h_1, h_2, h_3, h_4)\,,$$

so geometrically $Dq(a, b, c)$ is a quaternion perpendicular to a, b, c with the magnitude of the parallelepiped with edges a, b, c. (This aspect is analogous to the cross product in vector algebra.)

3. Regular Functions

In quaternionic analysis, unlike complex analysis, the notions of holomorphicity, analyticity, harmonicity, and conformality do not coincide. In this section, however, we provide analogues to these ideas in the context of quaternions and introduce regular functions, the main subjects of our analysis.

Definition 3.1 (Holomorphicity)**.** A quaternionic function f is quaternion-differentiable on the left at a point q if the limit

$$\frac{df}{dq} = \lim_{h\to 0} \frac{f(q+h) - f(q)}{h}$$

exists as $h \to 0$ from any direction in $\mathbb{H}$.

This strict definition is not very useful, since even simple functions like q^2 are directionally dependent. Indeed, the only quaternion-differentiable functions are linear, although the proof is somewhat involved and beyond the scope of this paper.

Proposition 3.2. *If a function is quaternion-differentiable on a connected open set U, then on U it must be of the form $f(q) = a + qb$ where $a, b \in \mathbb{H}$.*

Proof. See [10]. ■

Definition 3.3. A quaternionic monomial is a function $f : \mathbb{H} \to \mathbb{H}$ of the form

$$f(q) = \Pi_{i=0}^{r} a_i q$$

where r is the degree of the monomial and $a_0, \ldots, a_r$ are constant quaternions.

A quaternionic polynomial is then a finite sum of quaternionic monomials.

Proposition 3.4. *This notion of analyticity is equivalent to the notion of a function being real-analytic in 4 real variables.*

Proof. We can write any polynomial function f as

$$f(q) = f_0(w, x, y, z) + \sum_i f_i(w, x, y, z) e_i$$

where f_0 and f_i are four real-valued polynomials in the four real variables w, x, y, z. But

$$\left.\begin{array}{rl} w & = \frac{1}{4}(q - iqi - jqj - kqk), \\ x & = \frac{1}{4i}(q - iqi + jqj + kqk), \\ y & = \frac{1}{4j}(q + iqi - jqj + kqk), \\ z & = \frac{1}{4k}(q + iqi + jqj - kqk). \end{array}\right\}$$

By substituting these expressions for w, x, y, z in the polynomials f_0, f_i, we get $f(q)$ as a sum of expressions in q that are quaternionic monomials, so f is a quaternionic polynomial. ■

Definition 3.5 (Harmonicity)**.** Harmonic functions are those which satisfy Laplace's equation, which for a function on $\mathbb{R}^n$ is

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

In the context of quaternions, this means that a function $f(q)$ satisfies the equation

$$\frac{\partial^2 f}{\partial a^2} + \frac{\partial^2 f}{\partial b^2} + \frac{\partial^2 f}{\partial c^2} + \frac{\partial^2 f}{\partial d^2} = \Delta f = 0.$$

In complex analysis, real and imaginary parts of holomorphic functions are harmonic, but this correspondence does not exist in quaternionic analysis. Nevertheless, harmonic functions are useful for constructing regular functions.

Quaternionic analysis brings with it from complex analysis the geometric concept of conformality, however.

Definition 3.6 (Conformality)**.** Conformal mappings in quaternionic analysis are the same as those in complex analysis; they are mappings which are locally angle-preserving.

Theorem 3.7. *The conformal group (or group of all conformal mappings) of $\mathbb{H}$, and consists of functions of the form*

$$f(q) = (aq + b)(cq + d)^{-1}.$$

Proof. We let C be the group of orientation-preserving conformal mappings of $\mathbb{H}^*$, and let D be the set of mappings of the form above. Then if $f \in D$, f has differential

$$df_q = \left(ac^{-1}d - b\right)(cq+d)^{-1}cdq(cq+d)^{-1}.$$

This is of the form $\alpha dq\beta$, which is a dilatation and a rotation combined, so f is conformal and orientation-preserving. Therefore, D is a subset of C. Now C is generated by rotations, dilatations, translations and the inversion in the unit sphere followed by a reflection, i.e. by the mappings $q \mapsto \alpha q\beta, q \mapsto q+\gamma$ and $q \mapsto q^{-1}$. If a mapping in D is followed by any of these mappings, it remains in D; hence $CD \subseteq D$. It follows that $D = C$. ∎

Definition 3.8. We define the left and right Cauchy–Fueter equations as follows:

$$\frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial a} + i\frac{\partial f}{\partial b} + j\frac{\partial f}{\partial c} + k\frac{\partial f}{\partial d},$$

$$\frac{\partial_r f}{\partial \bar{q}} = \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}i + \frac{\partial f}{\partial c}j + \frac{\partial f}{\partial d}k.$$

Definition 3.9 (Regular function)**.** The main functions of interest in quaternionic analysis are regular functions. A quaternionic function is left- or right-regular if

$$\frac{\partial_l f}{\partial \bar{q}} = 0 \text{ or } \frac{\partial_r f}{\partial \bar{q}} = 0,$$

respectively.

These functions are harmonic and real-analytic, as we will show, but sadly not holomorphic or conformal. Counterintuitively, not even the identity function is regular. As we will see, one of the key motivations for this looser definition is we can adapt the Cauchy formula to quaternions.

4. Constructing Regular Functions

There are two ways to construct regular functions from real harmonic functions, the first being through the differential operator $\partial_l f$ and the second by considering it as, locally, the real part of a regular function.

Definition 4.1. The differential operator $\partial_l f$ is defined as

$$\partial_\ell f = \frac{1}{2}\bar{\Gamma}_r(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} - e_i\frac{\partial f}{\partial x_i}\right)$$

where Γ_r is the map

$$\Gamma_r(df) = \frac{\partial f}{\partial a} + i\frac{\partial f}{\partial b} + j\frac{\partial f}{\partial c} + k\frac{\partial f}{\partial d}.$$

Theorem 4.2. *If f is a harmonic real-valued function, then $\partial_l f$ is regular.*

Proof. A full explanation is given in [10], following from $\Delta = 4\bar{\partial}_\ell\partial_\ell$. ∎

Theorem 4.3. *If u is a harmonic real-valued twice-differentiable function defined on an star-shaped open set $U \in \mathbb{H}$, there exists a regular function f on U such that $\Re f = u$.*

Proof. Without loss of generality, we assume that U contains the origin and is star-shaped with respect to it, since we can easily adjust this for any other points. Thus we will show that the function

$$f(q) = u(q) + 2\mathrm{Pu}\int_0^1 s^2 \partial_\ell s u(sq) q \ ds,$$

where Pu represents the pure quaternion (i.e. non-real) part, is regular in U. Since

$$\begin{aligned}\operatorname{Re}\int_0^1 s^2 \partial_\ell u(sq) q ds &= \frac{1}{2}\int_0^1 s^2 \left\{ t\frac{\partial u}{\partial t}(sq) + x_i \frac{\partial u}{\partial x_i}(sq)\right\} ds \\ &= \frac{1}{2}\int_0^1 s^2 \frac{d}{ds}[u(sq)] ds \\ &= \frac{1}{2}u(q) - \int_0^1 s u(sq) ds,\end{aligned}$$

we have

$$f(q) = 2\int_0^1 s^2 \partial_\ell u(sq) q ds + 2\int_0^1 s u(sq) ds.$$

And because u and $\partial_\ell u$ have continuous partial derivatives in U, we can then differentiate under the integral sign to obtain for $q \in U$

$$\bar{\partial}_\ell f(q) = 2\int_0^1 s^2 \bar{\partial}_\ell \left[\partial_\ell u(sq)\right] q ds + \int_0^1 s^2 \left\{\partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i\right\} ds + 2\int_0^1 s^2 \bar{\partial}_\ell u(sq) ds.$$

However, $\bar{\partial}_\ell \left[\partial_\ell u(sq)\right] = \frac{1}{4} s \Delta u(sq) = 0$ since u is harmonic in U, so

$$\begin{aligned}\partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i &= -2\overline{\partial_\ell u(sq)} \\ &= -2\bar{\partial}_\ell u(sq).\end{aligned}$$

Thus $\bar{\partial}_\ell f = 0$ in U and so f is regular. ∎

We can also construct regular functions from analytic complex functions.

Definition 4.4. For $q \in \mathbb{H}$. define $n_q : \mathbb{C} \to \mathbb{H}$ to be the embedding of the complex numbers in the quaternions so that q is the image of a complex number $\zeta(q)$ lying in the upper half-plane:

$$\eta_q(x + iy) = x + \frac{\mathrm{Pu}\, q}{|\mathrm{P}_u q|} y,$$

$$\zeta(q) = \mathrm{Re} + i|\,\mathrm{Pu}\, q|.$$

Theorem 4.5. *For a complex function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on an open set $U \in \mathbb{C}$, we define $\bar{f} : \mathbb{H} \to \mathbb{H}$ to be*

$$\bar{f}(q) = \eta_q \circ f \circ \zeta(q),$$

and $\Delta \bar{f}$ is regular on $\zeta^{-1}(U)$.

Proof. We write $r = \mathrm{Pu} q$ and $u(x,y), v(x,y)$ for the real and imaginary parts respectively of $f(x+yi)$. This gives

$$\begin{aligned}
\mathrm{Re}\left[\bar{\partial}_\ell \bar{f}(q)\right] &= \frac{1}{2}\left\{\frac{\partial}{\partial w}[u(w,|r|)] - \Box \cdot \left[\frac{r}{|r|} v(w,|r|)\right]\right\} - \frac{1}{2}u_1(w,|r|) - \frac{v(w,|r|)}{|r|} - \frac{1}{2}v_2(w,|r|) \\
\mathrm{Pu}\left[\bar{\partial}_\ell \bar{f}(q)\right] &= \frac{1}{2}\left\{\Box[u(w,|r|)] + \frac{\partial}{\partial t}\left[\frac{r}{|r|}v(w,|r|)\right] + \Box \times \left[\frac{r}{|r|}v(w,|r|)\right]\right\} \\
&= \frac{1}{2}\left\{\frac{r}{|r|}u_2(w,|r|) + \frac{r}{|r|}v_1(w,|r|)\right\}
\end{aligned}$$

Because f is analytic, it obviously must satisfy the Cauchy-Riemann equations, meaning that $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, so

$$\bar{\partial}_\ell f = -\frac{v(w,|r|)}{|r|}.$$

Then,

$$\begin{aligned}
\bar{\partial}_\ell \Delta \bar{f}(q) = \Delta \bar{\partial}_\ell f(q) &= -\left(\frac{\partial^2}{\partial w^2} + \frac{1}{r}\frac{\partial^2}{\partial r^2} r\right)\frac{v(w,r)}{r} \\
&= -\frac{v_{11}(w,r) + v_{22}(w,r)}{r} \\
&= 0,
\end{aligned}$$

so $\Delta \bar{f}$ is regular on $\zeta^{-1}(U)$. ∎

5. Analogues to Integral and Series Results

Before presenting the quaternionic equivalents of the power series and Laurent series expansions, which were developed by Fueter, we must define some special functions to express them.

Definition 5.1. We define ν as an unordered set of n integers, each between 1 and 3; the set of all ν for a given n is denoted by σ_n. Then the nth order differential operator ∂_ν is

$$\partial_\nu = \frac{\partial^n}{\partial a^{n_1} \partial b^{n_2} \partial c^{n_3}}.$$

Definition 5.2. Then, the functions $G_\nu(q)$ and $P_\nu(q)$ are defined as follows:

$$G_\nu(q) = \partial_\nu G(q)$$

and

$$P_\nu(q) = \frac{1}{n!}\sum (a e_{i_1} - x_{i_1}) \ldots (a e_{i_n} - x_{i_n}),$$

where $e_1, e_2, e_3 = i, j, k$ and $x_1, x_2, x_3 = b, c, d$ and $\nu = \{i_1, \ldots i_n\}$ and P_ν is the sum over all permutations of the $i_k's$.

Theorem 5.3 (Laurent series expansion)**.** *Given a function f which is regular on an open set U except possibly at one point q_0. Then, there is a neighborhood N of q_0 such that if $q \in N$, $f(q)$ can be represented by the series*

$$f(q)=\sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n}\left\{P_\nu\left(q-q_0\right)a_\nu+G_\nu\left(q-q_0\right)b_\nu\right\}$$

which converges in an annulus

$$\{q \mid r \leq |q-q_0| \leq R\} \in N.$$

The coefficients a_ν and b_ν are given by

$$a_\nu=\frac{1}{2\pi^2}\int_C G_\nu\left(q-q_0\right)Dqf(q),$$

$$b_\nu=\frac{1}{2\pi^2}\int_C P_\nu\left(q-q_0\right)Dqf(q).$$

where C is any closed 3-chain in $U\setminus\{q_0\}$ which is homologous to ∂B for some ball B with $q_0 \in B \subset U$ (so that C has wrapping number 1 about q_0).

Proof. The proof of this is long and technical and thus outside the scope of this paper, but can be found in [10]. ■

Theorem 5.4 (Cauchy's theorem)**.** *If f is regular on every point of a 4-parallelepiped C,*

$$\int_{\partial C} Dq\, f=0.$$

Corollary 5.5. *If f is a right-regular function and g is a left-regular function, we have that*

$$\int_{\partial\sigma} f d\boldsymbol{Q} g=0.$$

Theorem 5.6. *For a hypersurface $\partial\sigma$ in $\mathbb{E}^4$ with $q_0 \in \partial\sigma$, we have*

$$\int_{\partial\sigma}\Delta\left(q-q_0\right)^{-1}dQ=8\pi^2.$$

Proof. See [10] and [1]. Note that for the surface element of a sphere with radius $|q|$ in four-dimensional Euclidean space, we have

$$dq=|q|^2 qdS$$

where dS is the surface element of the corresponding unit square. ■

Now we can move to our main result, the Cauchy–Fueter integral formula.

Theorem 5.7 (Cauchy–Fueter integral formula)**.** *If f is regular on every point of a (positively oriented) 4-parallelepiped C and q_0 is a point within C,*

$$f\left(q_0\right)=\frac{1}{2\pi^2}\int_{\partial C}\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dq\, f(q).$$

Proof. When ε is small enough, the hypersurface centered at q_0 defined by $|q-q_0|=\varepsilon$ lies inside $\partial\sigma$. In the region between the surface of the ε -sphere and $\partial\sigma$ both F and $\Delta_4\left(\mathrm{q}-\mathrm{q}_0\right)^{-1}$ are regular so that, using 5.5 we can show that

$$\frac{1}{8\pi^2}\int_{\partial\sigma} F(q)dQ\Delta_4\left(q-q_0\right)^{-1}=\frac{1}{8\pi^2}\int_{|q-q_0|=\varepsilon} F(q)dQ\Delta_4\left(q-q_0\right)^{-1}.$$

We can find the surface element for the last integral by replacing q with $q-q_0$ in the equation from the proof of 5.6, giving

$$dq=\left|q-q_0\right|^2\left(q-q_0\right)dS.$$

The function F has to be sufficiently differentiable so that

$$F(q)=F\left(q_0\right)+O\left(\left|q-q_0\right|\right)$$

as $|q-q_0|\to 0$. Therefore, the limit of the last integral as $\varepsilon\to 0$ is

$$\begin{aligned}&\lim\frac{4}{8\pi^2}\int_{|q-q_0|=1} F(q)\varepsilon^2\left(q-q_0\right)\varepsilon^{-2}\left(q-q_0\right)^{-1}dS\\&=\lim\frac{1}{2\pi^2}\int_{|q-q_0|=1}\left(F\left(q_0\right)+O(\varepsilon)\right)dS=F\left(q_0\right).\end{aligned}$$

■

Note that the elementary potential function in $\mathbb{R}^4$ is q^{-2}, analogous to the complex logarithm. Its Frechet derivative is $G(q)=\frac{q^{-1}}{|q|^2}$, the key function in the integral formula, just as the derivative of the logarithm is z^{-1}.

Many other fundamental results from complex analysis transfer over as a consequence, including the maximum modulus principle and Liouville's theorem. The proofs are extremely similar to their complex analogues.

Corollary 5.8 (Maximum modulus principle)**.** *Suppose that $U\subset\mathbb{H}$ is a connected open set and $f:U\to\mathbb{H}$ is regular. If there exists some z_0 so that $|f(z_0)|\geq|f(z)|$ for all $z\in U$, then f is constant.*

Corollary 5.9 (Liouville's theorem)**.** *A bounded regular function is constant.*

6. Applications

Quaternions and quaternionic functions have a range of applications, though not quite as commonplace as complex numbers. They were initially often used for rotations but displaced by vector algebra, though because of their computational efficiency have had a resurgence recently. They are also sometimes useful in physics, where important equations can be expressed concisely with them.

The higher dimensions of quaternions makes them convenient for representing 3D geometry, spatial rotations in particular. Rotating by an angle θ around an axis defined by the unit vector

$$\vec{u}=(u_x,u_y,u_z)=u_xi+u_yj+u_zk$$

can be expressed by

$$q=e^{\frac{\theta}{2}(u_xi+u_yj+u_zk)}=\cos\frac{\theta}{2}+(u_xi+u_yj+u_zk)\sin\frac{\theta}{2},$$

based on Euler's formula, thus expressing a single axis and the rotation. [6] These are more compact than matrices and easier to compose, and by using four variables they avoid the issue of losing one degree of freedom at certain configurations where axes align (known as gimbal lock) like with three Euler angles. [8]

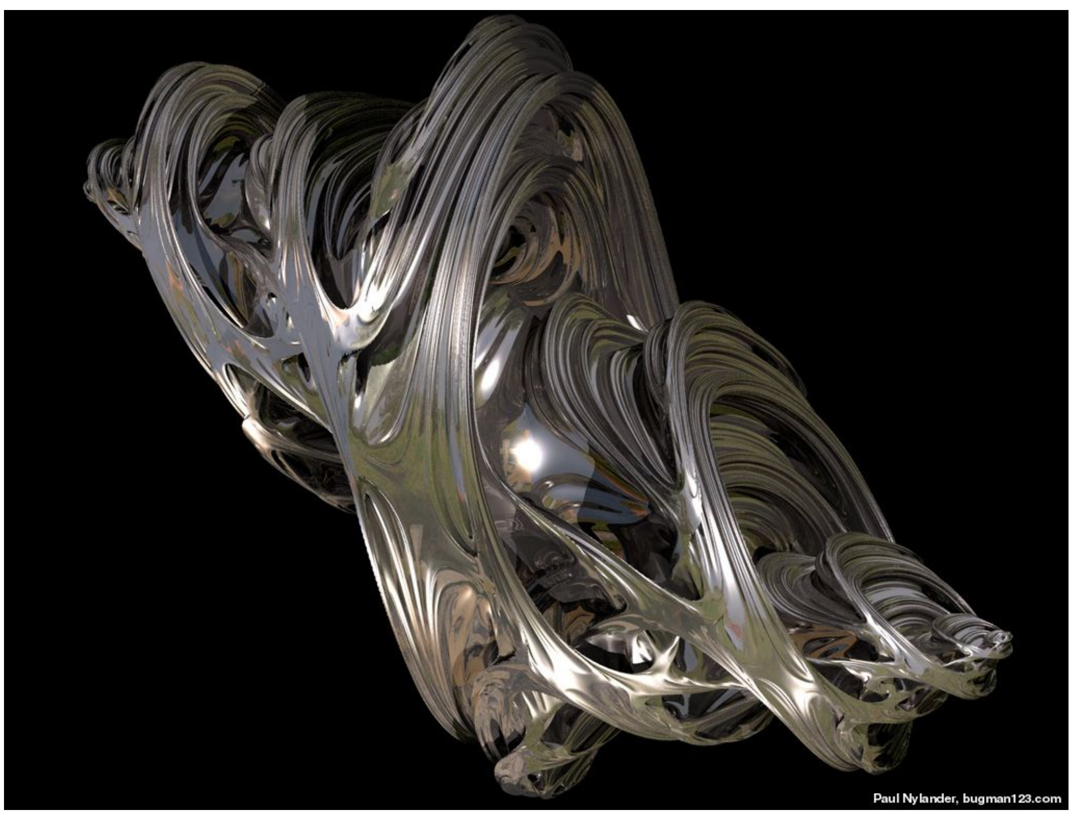
Over time, quaternions have attracted strong interest from physics, because their symmetries and unusual properties may encapsulate real-world behavior. Relativity's Lorentz transforms and Minkowski space find elegant correspondences in quaternionic analysis, although the physics required is beyond the scope of this paper. A detailed treatise may be found in [2], but we can demonstrate here the relatively less technical example of Maxwell's equations:

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla \cdot \mathbf{E} = \rho$$
$$\frac{1}{c}\frac{\partial \mathrm{H}}{\partial t} + \nabla \times \mathbf{E} = 0$$
$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H}$$

can be expressed with the operator $\Box^* = -\frac{i}{c}\frac{\delta}{\delta t} + \nabla$ as

$$\Box^*(\mathbf{E} + i\mathbf{H}) = -\rho + \frac{i}{c}\mathbf{J}.$$

To close our paper, here is the taffy-like result of extending the Julia set to quaternions:



References

[1] Cipher A Deavours. "The quaternion calculus". In: *The American Mathematical Monthly* 80.9 (1973), pp. 995–1008.

[2] Igor Frenkel and Matvei Libine. "Quaternionic analysis, representation theory and physics". In: *Advances in Mathematics* 218.6 (2008), pp. 1806–1877.

[3] Herrn Frobenius. "Über lineare Substitutionen und bilineare Formen". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 1878.84 (1878), pp. 1–63.

[4] Rud Fueter. "Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen". In: *Commentarii Mathematici Helvetici* 8.1 (1935), pp. 371–378.

[5] William Rowan Hamilton. *Elements of quaternions.* Longmans, Green, & Company, 1866.

[6] Yan-Bin Jia. *Quaternions and rotations.* 2013.

[7] John Kronsbein. "Kinematics—Quaternions—Spinors—and Pauli's Spin Matrices". In: *American Journal of Physics* 35.4 (1967), pp. 335–342.

[8] Logah Perumal. "Quaternion and its application in rotation using sets of regions". In: *International Journal of Engineering and Technology Innovation* 1.1 (2011), p. 35.

[9] RD Schafer. "An Introduction to Nonassociative Algebras". In: *American Mathematical Society* 61 (1955), pp. 469–484.

[10] Anthony Sudbery. "Quaternionic analysis". In: *Math. Proc. Camb. Phil. Soc.* Vol. 85. 2. 1979, pp. 199–225.